Separability and entanglement of identical bosonic systems

2006 J. Phys. A: Math. Gen. 39 L555

(http://iopscience.iop.org/0305-4470/39/36/L01)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) L555–L557 doi:10.1088/0305-4470/39/36/L01

LETTER TO THE EDITOR

Separability and entanglement of identical bosonic systems

Xiao-Hong Wang[1], **Shao-Ming Fei**[1,2] **and Ke Wu**[1]

[1] Department of Mathematics, Capital Normal University, Beijing, People's Republic of China
[2] Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany

Received 26 October 2005, in final form 11 November 2005
Published 18 August 2006
Online at stacks.iop.org/JPhysA/39/L555

Abstract

We investigate the separability of arbitrary n-dimensional multipartite identical bosonic systems. An explicit relation between the dimension and the separability is presented. In particular, for $n = 3$, it is shown that the property of PPT (positive partial transpose) and the separability are equivalent for tripartite systems.

PACS numbers: 03.67.Hk, 03.65.Ta, 89.70.+c

Quantum entanglement plays essential roles in quantum information processing and quantum computation. The entangled states provide key resources for a vast variety of novel phenomena such as quantum cryptography, quantum teleportation, super dense coding, etc [1]. An important problem in the theory of quantum entanglement is the separability. One of the famous separability criteria was given by Peres [2]. It says that all separable states necessarily have a positive partial transpose (PPT), which is further shown to be also sufficient for states on $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^2 \otimes \mathbb{C}^3$ [3, 4], where $\mathbb{C}^n$ denotes the n-dimensional complex space. There have been many results on the separability and entanglements of mixed states, see e.g., [5–9]. In particular, it is shown that every quantum states ρ supported on $\mathbb{C}^M \otimes \mathbb{C}^N$, $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^N$ and $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^N$ with positive partial transposes and rank $r(\rho) \leqslant N$ are separable and have a canonical form [5–7].

Although the entanglement is extensively studied for distinguishable particle systems, the entanglement of identical particle systems has been less investigated. In fact in certain systems such as quantum dots [10], Bose–Einstein condensates [11] and parametric down conversion [12], the entanglement should be treated as the one of identical particle systems. Schliemann *et al* [10, 13] have discussed the entanglement in two-fermion systems. They found that the entanglement in two-fermion systems is analogous to that in a two-distinguishable particle system. The results for two-boson systems are quite different. Li *et al* [14] and Paskauskas and You [15] have studied this problem of two-boson systems. For multipartite bosonic systems, there are very few discussions. Recently, the author in [16] obtained the canonical form for pure states of three identical bosons and classified the entanglement correlation into two types, the analogous GHZ and the W-types. In [17], it has been shown that rank n and rank $\frac{n(n+1)}{2}-2$ PPT bosonic mixed states in the symmetrized tensor product space $\mathcal{S}(\mathbb{C}^n \otimes \mathbb{C}^n)$ are separable, and all three-qubit ($n = 2$) bosonic PPT states are separable as well. For bosonic mixed state ρ in a k-qubit system, $k \geqslant 4$, ρ is PPT, which implies that ρ is separable, except for the case of maximal rank.

In this letter, we investigate the separability of multipartite identical bosonic systems with arbitrary dimension n. Let $\mathcal{H} = \mathcal{S}(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n)$ denote the symmetrized tensor product space of k n-dimensional spaces associated with Alice, Bob, Charle, etc. The dimension of the space $\mathcal{H}$ is given by [18]

$$I_n^k = \frac{(n+k-1)!}{k!(n-1)!} = C_{n+k-1}^k. \tag{1}$$

We first consider the case of $k = 3$.

Theorem 1. *Let ρ be a bosonic mixed state in $\mathcal{S}(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n)$, with a positive partial transpose with respect to Alice. If the rank of ρ, $r(\rho) \leqslant n^2$, then ρ is separable.*

Proof. We first prove the case of $n = 3$. Suppose that the state ρ is a PPT state with respect to Alice and has a rank 9. We can treat it as a bipartite PPT state in a 3×9 dimensional space of Alice–(Bob,Charlie). From theorem 1 in [5] (also theorem 1 in [6]), such a state of rank 9 is necessarily separable and can be represented as $\rho = \sum_{i=1}^{9} p_i |e_i, \Psi_i\rangle\langle e_i, \Psi_i|$, where the vectors $|\Psi_i\rangle$ are generally entangled pure states associated with the spaces of Bob and Charlie. As $|\Psi_i\rangle$ are mutually orthogonal, they belong to the range of the reduced density matrix (partial trace with respect to the space associated with Alice) $\mathrm{Tr}_A \rho$, and hence $|\Psi_i\rangle \in \mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3)$. Moreover $|e_i, \Psi_i\rangle$ belong to the range of ρ. Therefore $|e_i, \Psi_i\rangle \in \mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$. According to Schmidt decomposition we can write $|\Psi_i\rangle = a_i|00\rangle + b_i|11\rangle + c_i|22\rangle$ for some $a_i, b_i, c_i \in \mathbb{C}$, where $|0\rangle, |1\rangle, |2\rangle$ are the Schmidt basic vectors in $\mathbb{C}^3$. The only possible forms of $|e_i, \Psi_i\rangle$ satisfying the above conditions are $|000\rangle, |111\rangle$ or $|222\rangle$. Therefore ρ is separable.

When the rank of ρ is strictly less than 9, ρ can be embedded into a smaller space. For instance, if $r(\rho) = 8$, ρ is supported on spaces 2×8 or 3×8. ρ is then separable in the partition Alice–(Bob,Charlie) and can be again written as $\rho = \sum_{i=1}^{8} p_i |e_i, \Psi_i\rangle\langle e_i, \Psi_i|$. By using the same procedure as above, we can prove that $|e_i, \Psi_i\rangle$ is fully separable, and hence ρ is separable. The general n-dimensional case can be proved similarly. $\square$

Remark 1. From the theorem we see that a bosonic mixed state ρ in $\mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$ with a positive partial transpose is separable if $r(\rho) \leqslant 9$. As the dimension of the space of $\mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$ is 10, theorem 1 says that almost all the PPT bosonic mixed states in $\mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$ are separable, except for the case $r(\rho) = 10$. Hence the rank of a bound entangled state in $\mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$ has to be 10.

When $n = 4$, we have $I_4^3 = 20$. As ρ is separable if $r(\rho) \leqslant 16$, all bound entangled states ρ in $\mathcal{S}(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n)$ satisfy $17 \leqslant r(\rho) \leqslant 20$.

Theorem 2. *Let ρ be a PPT bosonic mixed state in $\mathcal{S}(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n)$ with k subsystems ($k \geqslant 4$). If $r(\rho) \leqslant I_n^{k-1}$, then ρ is separable.*

Proof. We prove the case of $n = 3$ (the other cases can be proved similarly). Assume that ρ is PPT, say with respect to the space associated with Alice, with rank $I_3^{k-1} = \frac{k(k+1)}{2}$.

If we consider ρ as a bipartite state in the partition Alice–the rest, ρ is supported on $\mathbb{C}^3 \otimes \mathcal{S}((\mathbb{C}^3)^{\otimes k-1})$. From [5], ρ is separable with respect to this partition and has a form,

$\rho = \sum_{i=1}^{\frac{k(k+1)}{2}} p_i |e_i, \Psi_i\rangle\langle e_i, \Psi_i|$, where $|e_i\rangle$ (resp. $|\Psi_i\rangle$) are vectors on the spaces associated with Alice (resp. the rest).

We prove result by induction. We illustrate the procedure by proving the case of $k = 4$. As $|\Psi_i\rangle$ belong to the range of the reduced density matrix $\mathrm{Tr}_A\rho$, they must belong to $\mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$. Since ρ is PPT, $|\Psi_i\rangle\langle\Psi_i|$ is a PPT state in $\mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$. However, the rank $r(|\Psi_i\rangle\langle\Psi_i|) = 1$, from theorem 1, $|\Psi_i\rangle$ is separable, and can be written as $|\Psi_i\rangle = |f_i, f_i, f_i\rangle$ for some vectors $|f_i\rangle$ in $\mathbb{C}^3$. While the vectors $|e_i, \Psi_i\rangle$ belong to the range of ρ and hence $|e_i, \Psi_i\rangle \in \mathcal{S}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$. Therefore the only possible forms of $|e_i, \Psi_i\rangle$ are $|f_i, f_i, f_i, f_i\rangle$. Therefore ρ is separable. □

We have presented some separability criteria for multipartite bosonic mixed states. For tripartite PPT states, all bound entangled states have necessarily rank greater than n^2. For general multipartite PPT bosonic states with k subsystems ($k \geqslant 4$), if $r(\rho) \leqslant I_n^{k-1}$, ρ is separable. The results can be used to construct possible bound entangled states of identical bosonic systems. For instance, if $k = 4, n = 3$, we have $I_3^4 = 15$. The rank of a bound entangled state has to be between $I_3^3 = 10$ and 15.

Acknowledgment

The work is supported by Beijing Municipal Education Commission (no. KM 200510028021), National Natural Science Foundation of China (no. 10375038 and 90403018) and NKBRPC (2004-CB 318000).

References

[1] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[2] Peres A 1996 *Phys. Rev. Lett.* **77** 1413
[3] Horodecki M, Horodecki P and Horodecki R 1996 *Phys. Lett.* A **223** 1
[4] Horodecki P 1997 *Phys. Lett.* A **232** 333
[5] Horodecki P, Lewenstein M, Vidal G and Cirac I 2000 *Phys. Rev.* A **62** 032310
[6] Karnas S and Lewenstein M 2001 *Phys. Rev.* A **64** 042313
[7] Fei S M, Gao X H, Wang X H, Wang Z X and Wu K 2003 *Phys. Rev.* A **68** 022315
[8] Albeverio S, Fei S M and Goswami D 2001 *Phys. Lett.* A **286** 91
Albeverio S and Fei S M 2001 *J. Opt. B: Quantum Semiclass. Opt.* **3** 223
Fei S M, Gao X H, Wang X H, Wang Z X and Wu K 2002 *Phys. Lett.* A **300** 559
Fei S M, Gao X H, Wang X H, Wang Z X and Wu K 2003 *Int. J. Quantum Inform.* **1** 37
[9] Chen K, Albeverio S and Fei S M 2005 *Phys. Rev. Lett.* **95** 040504
[10] Schliemann J, Loss D and MacDonald A H 2001 *Phys. Rev.* B **63** 085311
[11] Sørensen A, Duan L M, Cirac J I and Zoller P 2001 *Nature* **409** 63
[12] Kim Y H, Chekhova M V, Kulik S P, Rubin M H and Shih Y 2001 *Phys. Rev.* A **63** 062301
[13] Schliemann J, Cirac J I, Kuś M, Lewenstein M and Loss D 2001 *Phys. Rev.* A **64** 022303
[14] Li Y S, Zeng B, Liu X S and Long G L 2001 *Phys. Rev.* A **64** 054302
[15] Paskauskas R and You L 2001 *Phys. Rev.* A **64** 042310
[16] Zeng B, Zhou D L, Xu Z and You L 2005 *Phys. Rev.* A **71** 042317
[17] Eckert K, Schliemann J, Bruß D and Lewenstein M 2002 *Ann. Phys., NY* **299** 88
[18] Hamermesh M 1962 *Group Theory and its Application to Physical Problems* (Reading, MA: Addison-Wesley)